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Density of ϕ-Polynomials

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario,
London 72, Ontario, Canada

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Let ϕ be a real function continuous on the real line. Let $V_n(\phi)$ be the family of functions of the form

$$F(A, x) = \sum_{k=1}^{n} a_k \phi(a_{n+k} x), \qquad a_1, ..., a_{2n} \text{ real.}$$

The problem of approximation by $V_n(\phi)$ is a special case of that of approximation by a curve of functions or equivalently approximation by γ-polynomials [1–3]. A function in $V_n(\phi)$, for some n, is called a ϕ-polynomial. We are interested in determining sets of functions in which the set of ϕ-polynomials is dense, with respect to the uniform (Chebyshev) norm.

DEFINITION. A subset $\mathscr{G}$ of a set of functions $\mathscr{F}$ is fundamental in $\mathscr{F}$ if the set of (finite) linear combinations of functions in $\mathscr{G}$ is dense in $\mathscr{F}$.

THEOREM. *Let $\mathscr{F}$ be a set of functions defined on a finite interval $[\alpha, \beta]$ of the real line. Let ϕ have a Taylor series $\sum_{k=0}^{\infty} c_k x^k$, absolutely convergent in a closed neighborhood I of zero, such that the set of powers x^k with nonzero coefficient c_k is fundamental in $\mathscr{F}$. If $\{\gamma_k\}$ is an infinite sequence of nonzero numbers converging to zero, then the set $\{\phi(0), \phi(\gamma_1 x), \phi(\gamma_2 x),...\}$ is fundamental in $\mathscr{F}$.*

Proof. It suffices to show that each power x^n with a nonzero coefficient c_n can be uniformly approximated by linear combinations of $\{\phi(0), \phi(\gamma_1 x), \phi(\gamma_2 x),...\}$. We shall prove that for each $j > 0$ there is a sequence of linear combinations with the kth linear combination equal to $x^n + O(k^{-j})$ on $[\alpha, \beta]$. There exists a positive constant $\mu \leqslant 1$ such that $\mu x \in I$ for $x \in [\alpha, \beta]$. Assume without loss of generality that $|\gamma_i| \leqslant \mu$ for all i, then $\phi(\gamma_i x)$ has a Taylor series convergent on $[\alpha, \beta]$ for all i. Suppose $c_1 \neq 0$. Select γ_i with $|\gamma_i| \leqslant 1/k^j$ and $|\gamma_i| < \mu^3$.

$$\frac{1}{\gamma_i c_1}[\phi(\gamma_i x) - \phi(0)] = x + \frac{c_2}{c_1} x^2 \gamma_i + \frac{c_3}{c_1} x^3 \gamma_i^2 + \cdots$$

$$= x + \gamma_i \left(\frac{c_2}{c_1} x^2\right) + \frac{\gamma_i}{c_1} \sum_{k=3}^{\infty} c_k x^k \gamma_i^{k-2}.$$

Now for $x \in [\alpha, \beta]$,

$$\sum_{k=3}^{\infty} | c_k x^k \gamma_i^{k-2} | \leqslant \sum_{k=3}^{\infty} | c_k x^k | \, | \gamma_i |^{k/3} \leqslant \sum_{k=3}^{\infty} | c_k | \, | \mu x |^k < \infty.$$

Now let us suppose that our assertion is true for all powers up to and including the $m-1$st and *all* j. Let $c_m \neq 0$. Select γ_i such that $| \gamma_i | \leqslant 1/k^j$ and $| \gamma_i | \leqslant \mu^{m+2}$. There exists l such that $| \gamma_i | \geqslant 1/(k^{j+l})$.

$$\frac{1}{\gamma_i^m c_m}[\phi(\gamma_i x) - \phi(0)] = \frac{c_1}{c_m} x \gamma_i^{1-m} + \cdots + \frac{c_{m-1}}{c_m} x^{m-1} \gamma_i^{-1} + x^m$$

$$+ \frac{c_{m+1}}{c_m} x^{m+1} \gamma_i + \cdots .$$

The first $m-1$ terms of the right-hand side are either zero or can be approximated with error $O(k^{-j})$ by our induction hypothesis: for example we can approximate

$$\frac{c_1}{c_m} x \gamma_i^{1-m} \quad \text{by} \quad \frac{c_1}{c_m} \gamma_i^{1-m} \left[x + O\left(\frac{1}{(k^{m(j+l)})}\right)\right] = \frac{c_1}{c_m} \gamma_i^{1-m} x + O\left(\frac{1}{k^{l+j}}\right).$$

The $m+1$st terms and after are

$$\gamma_i \left(\frac{c_{m+1}}{c_m} x^{m+1}\right) + \frac{\gamma_i}{c_m} \sum_{k=m+2}^{\infty} c_k x^k \gamma_i^{k-m-1}.$$

Now for $x \in [\alpha, \beta]$,

$$\sum_{k=m+2}^{\infty} | c_k x^k \gamma_i^{k-m-1} | \leqslant \sum_{k=m+2}^{\infty} | c_k x^k | \, | \gamma_i |^{k/(m+2)} \leqslant \sum_{k=m+2}^{\infty} | c_k | \, | \mu x |^k < \infty.$$

APPLICATIONS

1. If all coefficients c_k of the series are nonzero, the set of powers with nonzero coefficients is fundamental in $C[\alpha, \beta]$.

2. If $\phi(0) = 0$ and all but the zeroth coefficient of the series are nonzero, then the set of powers with nonzero coefficients is fundamental in $C_0[\alpha, \beta]$, the set of continuous functions on $[\alpha, \beta]$ vanishing at zero.

3. If ϕ is even and all even coefficients of the series are nonzero, the set of powers with nonzero coefficients is fundamental in $C[0, \beta]$.

4. If ϕ is odd and all odd coefficients of the series are nonzero, the set of powers with nonzero coefficients are fundamental in $C_0[0, \beta]$.

The first is a consequence of the Weierstrass theorem. The second can be deduced from the first. The third and fourth are consequences of the Müntz theorem.

GENERALIZATIONS

More generally we may be given ϕ continuous on an interval J containing zero and given a finite interval $[\alpha, \beta]$. *A* (ϕ, J) polynomial on $[\alpha, \beta]$ is any finite linear combination of terms of the form $a\phi(bx)$ such that $bx \in J$ for all $x \in [\alpha, \beta]$. Let $\{\gamma_k\} \to 0$. Then there exists K such that for $k \geqslant K$, $\gamma_k x \in J$ for all $x \in [\alpha, \beta]$. It is trivial to extend the theorem to cover this case.

Let $\phi(0) = 0$ and all but the zeroth coefficient of the Taylor series for ϕ be nonzero. Let $\{\gamma_k\}$ be a nonzero sequence with limit 0. Let f be given then $f - f(0) \in C_0[\alpha, \beta]$ and can be uniformly approximated by linear combinations of $\{\phi(\gamma_1 x), \phi(\gamma_2 x),...\}$. Hence the set of linear combinations of $\{1, \phi(\gamma_1 x), \phi(\gamma_2 x),...\}$ is dense in $C[\alpha, \beta]$.

Similar arguments show that if ϕ is odd and all odd coefficients of the Taylor series are nonzero, the set of linear combinations of $\{1, \phi(\gamma_1 x), \phi(\gamma_2 x),...\}$ is dense in $C[0, \beta]$.

Let ϕ be odd with the odd coefficients of its Taylor expansion nonzero. Let ψ be even with the even coefficients of its Taylor expansion nonzero. A linear combination of terms of the type $\phi(ax)$ or $\psi(bx)$ is called a (ϕ, ψ)-polynomial. The set of (ϕ, ψ) polynomials is dense in $C[\alpha, \beta]$.

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